Stable Distributions: Easy Introduction Including Full Basic Mathematics

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This article explains the basics of using stable distributions and why they are used. They are often useful in modeling in natural sciences and in financial modeling.

1. Introduction

Sometimes one wants to model the process of a random variable throughout time.

If the value Y at time $T+1$ can be well described by the value at T plus some random change ΔY, then this random change could be modeled with a stable distribution. In the basic model discussed here, the change is assumed to be identically and independently distributed (iid).

We have $Y_{T+1} = Y_T + \Delta Y$

Often this random change ΔY is modeled as being normally distributed. The normal distribution is a special case of the class of stable distributions (also known as alpha stable distributions).

When using a normal distribution it is possible to cut the original time step size ΔT in halve $\Delta t = \frac{\Delta T}{2}$ (I'm using capital T and Y to denote the original changes and lower cases for the smaller steps). If we want keep the distribution for two of these smaller time steps added together the same as for one of the original larger time steps ΔT, then we must find a distribution for the smaller change Δy such that ΔY is distributed the same as the sum of two smaller changes Δy.

It turns out, that if ΔY is normally distributed, then the smaller changes Δy will also have to be normally distributed (but with different mean and variance). In fact, that holds for any stable distribution. For any other distribution, the two smaller changes will not follow the same distribution as the larger change. If you want to break up a random non-stable distribution into two random iid components that when summed up result in the original non-stable distribution, you have to do calculations to determine the distributions of the smaller components. If you then cut the smaller components in halve again, their smaller components will again follow another distribution.

Therefore, the big advantage of stable distributions is, that you can go smaller and smaller while retaining the same distribution (with changes in parameter, e.g. center, dispersion and others - see later) or go larger and larger while also retaining the same distribution. The smaller and smaller also works down to continuous modeling.

A process continuous in time with stable distributed changes is called Lévy Process. The most well known is arithmetic Brownian motion, where the increments are normally distributed. The Brownian motion is continues in time and also continuous in its path. All other Lévy processes can experience jumps in their paths.

Any sum of iid random variables will converge to a stable distributions, if the number of added random variables is increased. All distributions with finite variance converge to a normal distribution when summing them. If for a processes any smaller time period shall follow the same distribution as any larger time period (adjusted for mean, scale etc.) then its changes must be stable distributed.

2. Why Use Alpha Stable Distributions Instead of Others For Modeling

1. They are the only limit distributions of property normalized and centered partial sum processes for iid random variables (the same as explained in the introduction).

2. The class of stable distributions includes distributions with heavy tails, high peaks, strong skewness and therefore can model a broad range of random processes.

In finance, this helps to model the fat tails (extreme outcomes are likely) and asymmetric leptokurtic (large losses are more likely that large gains) behavior seen in empirical asset returns

3. Mathematical Definition of Stable Distributions

A stable distribution is any distribution that has the following property

$$c_1 X_1 + c_2 X_2 \overset{d}{=} b(c_1, c_2) X + a(c_1, c_2)$$

where X_1, X_2, X iid; c_1, c_2 non-negative; a, b real

You may be familiar with the following formula

X_1, X_2 iid: $Var(X_1 + X_2) = Var(X_1) + Var(X_2)$

For $c_1, c_2 = 1$ it follows that $b = \sqrt{2}$

However, for many stable distributions b is different.

How is that possible?

The answer lies in the concept of variance. All stable distributions other than the normal distribution have infinite variance and hence the formula does not apply to them (is infinite larger than infinite + infinite?).

Stable distributions are characterized by a parameter α (index of stability) and β (skeweness).

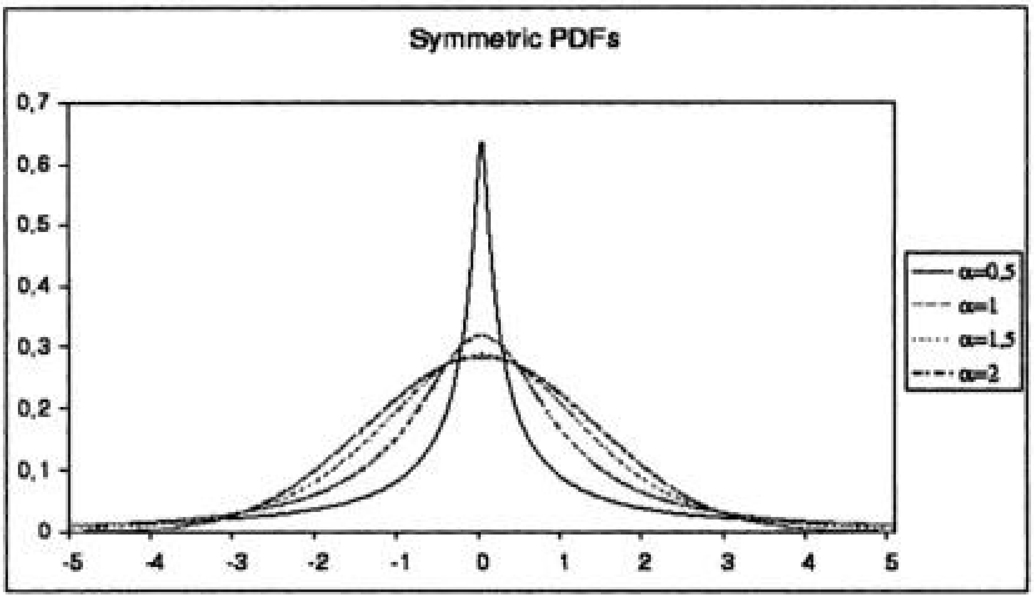
If $\alpha < 2$ then the variance is infinite

If $\alpha \leq 1$ then the mean does not exists (if you sample such a distribution, your sample mean will jump up and down heavily)

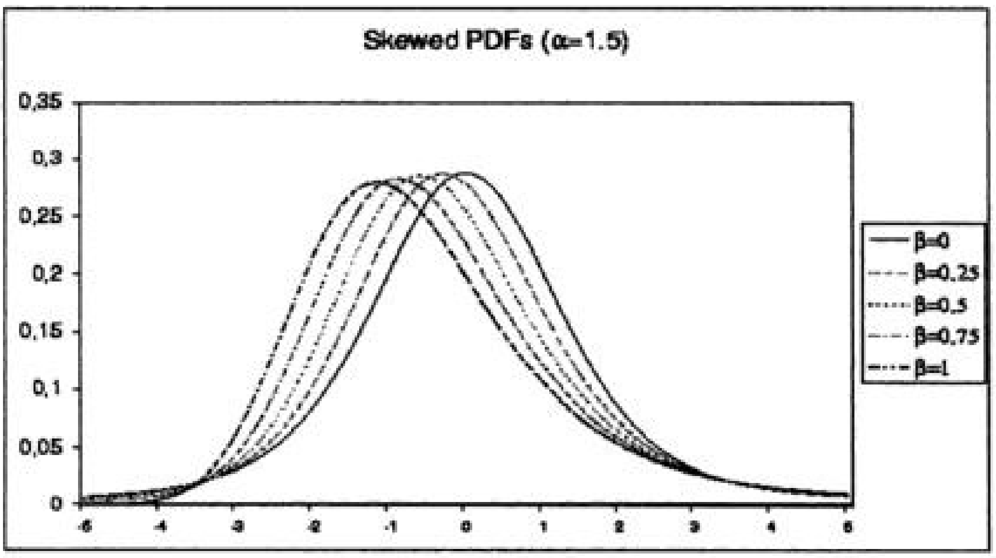
If $\alpha = 2$ then it's the Gaussian distribution

4. Graphic Examples of Alpha Stable Distributions

A few example probability density distribution functions for stable distributions are shown below. The next two graphics are from Stoyanov And Racheva-Jotova (2004).

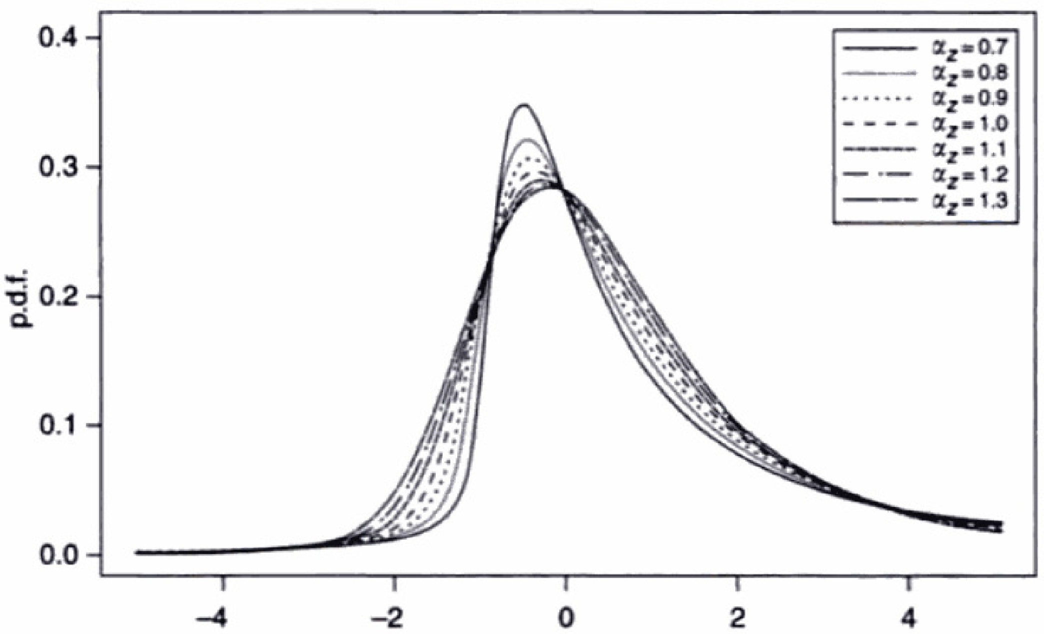


The probability density function for a standard symmetric alpha-stable random variable, varying the index of stability, alpha



Probability density functions for skewed stable random variables, varying the skewness parameter beta, for a given index of stability, alpha=1.5

The next image is from Los (2007; P.86). It shows skewed stable distributions.



Stable density in the Zolotarev $\mathbf{S}(\alpha_Z, \beta, \gamma, \delta; 0) = \mathbf{S}(\alpha_Z, 0.8, 1, 0; 0)$ parametrization.

5. The Density And Distribution Function 1

Unfortunately, the stable distribution in general does not have an explicit form for the density and distribution function (cannot be expressed mathematically; only exception is the normal distribution). So one has to work with the so called characteristic function. The characteristic function can be converted to approximate density and distributions functions (with numerical approximations).

$$\Phi_X(t) = \begin{cases} \exp(-\sigma^\alpha |t|^\alpha (1 - i\beta sign(t) \tan \frac{\pi\alpha}{2}) + i\mu t) & \alpha \neq 0 \\ \exp(-\sigma |t| (1 - i\beta \frac{2}{\pi} sign(t) \ln t) + i\mu t) & \alpha = 1 \end{cases}$$

Parameters:

$\alpha \in (0,2]$ Index of stability

$\beta \in [-1,1]$ Skewness parameter

$\mu \in \Re$ Shift parameter

$\sigma \in [0,\infty)$ Scale parameter

6. The Density And Distributions Function 2

From the characteristic function to approximate PDF

Definition Characteristic Function

$$\Phi_X(t) = E(e^{itX}) = \int_{-\infty}^{\infty} e^{itx}\, pdf(x) dx = E(\cos tX) + iE(\sin tX)$$

Theoretical Approach: Fourier Inversion Integral

$$pdf(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \Phi(t) dt$$

More (computationally) efficient approaches/approximations exist (cp. Stoyanov, Racheva-Jotova; 2004):

– Fast Fourier Transformation

– Infinite series expansions

– Polynomial-type approximations

– Integral representations of Zolotarev

– Approximations of McCulloch for symmetric stable distributions

7. 6 Properties of Stable Distributions

The following properties 1-6 are discussed by Stoyanov and Racheva-Jotova (2004).

Property 1

Let $X_1 \sim S_\alpha(\sigma_1, \beta_1, \mu_1)$ and $X_2 \sim S_\alpha(\sigma_2, \beta_2, \mu_2)$

then $X_1 + X_2 \sim S_\alpha(\sigma, \beta, \mu)$

with

$$\sigma = \left(\sigma_1^\alpha + \sigma_2^\alpha\right)^{1/\alpha} \quad \beta = \frac{\beta_1 \sigma_1^\alpha + \beta_2 \sigma_2^\alpha}{\sigma_1^\alpha + \sigma_2^\alpha} \quad \mu = \mu_1 + \mu_2$$

Property 1 says that if you add two stable distributed independent random variables with the same index of stability, the resulting sum will also be stable distributed and have the same index of stability.

Property 2

Let $X \sim S_\alpha(\sigma, \beta, \mu)$ and $a \in \Re$

Then $X + a \sim S_\alpha(\sigma, \beta, \mu + a)$

Property two identifies μ as a parameter that describes where the center of the distribution is.

Property 3

Let $X \sim S_\alpha(\sigma, \beta, \mu)$ and $a \in \Re$, $a \neq 0$

Then

$$aX \sim S_\alpha(|a|\sigma, sign(a)\beta, a\mu), \ \alpha \neq 1$$
$$aX \sim S_1\left(|a|\sigma, sign(a)\beta, a\mu - \frac{2}{\pi}(\ln(|a|)\sigma\beta)\right), \ \alpha = 1$$

Property 3 identifies σ as a dispersion parameter (knowing that μ is a shift parameter).

Property 4

For any $0 < \alpha < 2$ if $X \sim S_\alpha(\sigma, \beta, \mu)$

$$-X \sim S_\alpha(\sigma, -\beta, \mu)$$

Property 4 identifies β as a the parameter that shapes non symmetricity, which includes skewness.

Property 5

$X \sim S_\alpha(\sigma, \beta, \mu)$ is symmetric about μ if and only if $\beta = 0$

Property 6

Let $1 < \alpha \leq 2$ then μ equals the mathematical expectation of

$$X \sim S_\alpha(\sigma, \beta, \mu)$$

Property 7

Using Property 1,2, and 3 we get that if

$X, X_1, X_2, \ldots, X_n$ iid

with distribution

$S_\alpha(\sigma, \beta, \mu=0) \quad \alpha \neq 1$

then

$$X_1 + X_2 + \ldots + X_n \overset{d}{=} n^{1/\alpha} X$$

8. Risk Neutral Measure

For risk neutral measure see Sato (1999).

9. Practical Application

The simplest use of stable distributions is to construct a model with stable distributed changes and then calibrate the data (that shall be described) by using maximum likelihood estimation based on approximated probability density functions. In the further process Monte Carlo simulation might be used.

10. Further Reading, my Book Recommendations

There are 2 good books by mathematicians about stable distributions. They are both pure math books. Petrov's book (translated to English) contains many proofs of properties of stable distributions. The 2 books are:

- Petrov, V. V. (1995). *Limit Theorems of Probability Theory.* Oxford: Claredon.
- Sato, K. (1999). *Lévy Processes and Infinitely Divisible Distributions.* Cambridge: Cambridge University.

There are 2 good books on probability theory in general:

- Billingsley, P. (1986). *Probability and Measure (2nd Edition).* John Wiley & Sons: New York
- Lahiri, S. N., Athreya, K.B. (2006). *Measure Theory and Probability Theory.* New York: Springer.

References

- Los, C. A. (2007). Financial Market Risk. London: Taylor & Francis.
- Stoyanov, S., Racheva-Jotova, B. (2004). Numerical Methods for Stable Modeling in Financial Risk Management. In S. T. Rachev (Ed.), *Handbook of Computational and Numerical Methods in Finance.*(pp. 299-329). Boston: Birkhäuser.